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HW 4

Math 583B
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1. Derive the distribution for $\langle T'|\phi\rangle$ where

$$\langle T|\phi\rangle = \int_0^\infty (\ln x)\phi(x)\,dx \tag{1}$$

Solution.

$$\begin{aligned}
\langle T'|\phi\rangle = -\langle T|\phi'\rangle &= -\int_0^\infty \ln(x)\phi'(x)\,dx && (2)\\
&= -(\lim_{\epsilon\to 0}\int_\epsilon^1 \ln(x)(\phi(x)-\phi(0))'\,dx + \int_1^\infty \ln(x)\phi'(x)\,dx) && (3)\\
\text{Integrate by parts} & && (4)\\
&= -\lim_{\epsilon\to 0}\epsilon\ln(\epsilon)(\frac{\phi(\epsilon)-\phi(0)}{\epsilon}) + \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{\phi(x)-\phi(0)}{x}\,dx && (5)\\
&+ \int_1^\infty \frac{\phi(x)}{x}\,dx && (6)
\end{aligned}$$

$\epsilon\ln(\epsilon)\to 0$ while $\frac{\phi(\epsilon)-\phi(0)}{\epsilon}\to\phi'(0)$, which is finite, therefore

$$\langle T'|\phi\rangle = \int_0^1 \frac{\phi(x)-\phi(0)}{x}\,dx + \int_1^\infty \frac{\phi(x)}{x}\,dx \tag{7}$$

2. Prove Fourier transform relations for distributions

$$\frac{d\langle\hat{T}|\hat{\phi}\rangle}{dk} = ix\langle\hat{T}|\phi\rangle \qquad k\langle\hat{T}|\hat{\phi}\rangle = i\frac{d\langle\hat{T}|\phi\rangle}{dx} \tag{8}$$

Solution.

3. In $\mathcal{R}^2$, find the distribution $\Delta\ln|\vec{x}|$

Solution.

4. Consider IVPs of the form

$$\begin{aligned}
E_{tt}-\Delta E &= 0 && (9)\\
E(\vec{x},0) &= f(\vec{x}) && (10)\\
E_t(\vec{x},0) &= g(\vec{x}) && (11)
\end{aligned}$$

a.) Show that

$$u(\vec{x},t) = \frac{t}{4\pi}\int\int_{S^2} g(\vec{x}+t\vec{y})\,d\sigma \tag{12}$$

is a solution to the given IVP.

Solution.

b.) Show that $v = u_t$ solves the given IVP.

Solution.

c.) Show that the solution of the original IVP is

$$\frac{\partial}{\partial t}(\frac{t}{4\pi}\int\int_{S^2} f(\vec{x}+t\vec{y})\,d\sigma) + \frac{t}{4\pi}\int\int_{S^2} g(\vec{x}+t\vec{y})\,d\sigma \tag{13}$$

Solution.

5. Define distribution for locally integrable g

$$T_f' = j_a\delta(x-a) + j_b\delta(x-b) + T_{f'} \tag{14}$$

Solution.

6. Convergence of Fourier series in the sense of distributions
a.) Show that $Re\ln(1-e^{ix}) = \frac{1}{2}\ln(2(1-\cos x))$.

Solution.

b.) Show that

$$-\ln(|2\sin(\frac{x}{2})|) = \sum_{n=1}^{\infty}\frac{\cos nx}{n} \tag{15}$$

Solution.

c.) Derive distribution identities

Solution.

d.) Show that the convolution has the Fourier expansion

$$h(x) = \sum_{n=-\infty}^{\infty} i\pi\text{sgn}(n)\hat{f}(n)e^{inx} \tag{16}$$

Solution.

e.) Show that

$$f(x) = \frac{i}{\pi}\int_0^{2\pi} C(x-t)f(t)\,dt \tag{17}$$

and deduce the periodic Kramers-Kronig relations

Solution.